

REFLECTED BSDE OF WIENER-POISSON TYPE IN TIME-DEPENDENT DOMAINS

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ABSTRACT. In this paper we study multi-dimensional reflected backward stochastic differential equations driven by Wiener-Poisson type processes. We prove existence and uniqueness of solutions, with reflection in the inward spatial normal direction, in the setting of certain time-dependent domains.

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1. INTRODUCTION

Backward stochastic differential equations, BSDEs for short, is by now an established field of research. The solution to a classical BSDE, driven by a Wiener process W , is a pair of processes (Y, Z) such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

where ξ is a random variable that becomes known, with certainty, only at time T . In this setting $Y_t \in \mathbb{R}^d$, $d \geq 1$, and in the following we refer to the case $d = 1$ as the one-dimensional case and to the case $d > 1$ as the multi-dimensional case. Classical BSDEs have turned out important in many areas of mathematics including mathematical finance, see [EPQ] and the long list of references therein, stochastic control theory and stochastic game theory, see, e.g., [CK] and [HL], as well as in the connection to partial differential equations, see, e.g., [BBP] and [PP].

In [EKPPQ] a notion of *reflected* BSDE was introduced. A solution to a one-dimensional reflected BSDE is a triple of processes (Y, Z, Λ) satisfying

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) ds + \Lambda_T - \Lambda_t - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \\ Y_t &\geq S_t, \end{aligned}$$

where the barrier S is a given (one-dimensional) stochastic process. Λ is a continuous increasing process, with $\Lambda_0 = 0$, pushing the process Y upwards in order to keep it above the barrier. This is done with minimal energy in the sense that

$$\int_0^T (S_t - Y_t) d\Lambda_t = 0,$$

and consequently Λ increases only when Y is at the boundary of the space-time domain $\{(t, s) : s > S_t\}$. This type of reflected BSDE has important applications in the context of American options, optimal stopping and obstacle problem, see [EKPPQ], as well as in the context of stochastic game problems, see [CK].

In the multi-dimensional case there are at least two different types of reflected BSDEs studied in the literature.

The first type of multi-dimensional reflected BSDE was first studied in [GP] where the authors considered reflected BSDEs of the form

$$\begin{aligned} Y_t &= \xi + \int_t^T f(s, Y_s, Z_s) ds + \Lambda_T - \Lambda_t - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \\ (1.1) \quad Y_t &\in \Omega, \quad 0 \leq t \leq T, \end{aligned}$$

where $\Omega \subset \mathbb{R}^d$. In this case $\Lambda_0 = 0$ and

$$\begin{aligned} \Lambda_t &= \int_0^t \gamma_s d|\Lambda|_s, \quad \gamma_s \in N^1(Y_s), \\ (1.2) \quad d|\Lambda| &(\{t \in [0, T] : Y_t \in \Omega\}) = 0, \end{aligned}$$

where $N^1(Y_s)$ is the unit inner normal to Ω at Y_s . In particular, the process Λ_t is of bounded total variation $|\Lambda|$ and it increases only when Y is at the boundary of Ω . To be more precise, when Y is at the boundary it is pushed into the domain along $\gamma \in N^1(Y)$. In [GP] existence and uniqueness for this problem is established and we note that this problem, and its analysis, is inspired by and resemble the corresponding theory for reflected stochastic differential equations, see [T], [S], and [LS]. Naturally one can attempt, as in the case of reflected SDEs, to study this problem with oblique reflection instead of reflection in the direction of the inner unit normal. However, to the best of our knowledge the case of oblique reflection is a less developed area of research in the context of BSDEs and we are only aware

of the work in [R], where the author studies an obliquely reflected BSDE in an orthant.

The second type of multi-dimensional reflected BSDEs occurs in the study of optimal switching problems and stochastic games, see, e.g., [AF], [AH], [DHP], [HT], [HZ], and references therein. In the generic optimal switching problem a production facility is considered and it is assumed that the production can run in $d \geq 2$ different production modes. Furthermore, there is a stochastic process $X = (X_t)_{t \geq 0}$ which stands for the market price of the underlying commodities and other financial parameters that influence the production. When the facility is in mode i , the revenue per unit time is $f_i(t, X_t)$ and the cost of switching from mode i to mode j , at time t , is $c_{ij}(t, X_t)$. Let (Y_t^1, \dots, Y_t^d) be the value function associated with the optimal switching problem, on the time interval $[t, T]$, i.e., Y_t^i stands for the optimal expected profit if, at time t , the production is in mode i . In this case, one can prove, under various assumptions, see [AF], [AH], and [DHP], that (Y_t^1, \dots, Y_t^d) solves the reflected BSDE

$$\begin{aligned} Y_t^i &= \xi_i + \int_t^T f_i(s, X_s) ds - \int_t^T Z_s^i dW_s + \Lambda_T^i - \Lambda_t^i, \\ Y_t^i &\geq \max_{j \in A_i} \left(Y_t^j - c_{ij}(t, X_t) \right), \\ (1.3) \quad \int_0^T \left(Y_t^i - \max_{j \in A_i} \left(Y_t^j - c_{ij}(t, X_t) \right) \right) d\Lambda_t^i &= 0, \end{aligned}$$

where $i \in \{1, \dots, d\}$, $0 \leq t \leq T$, and $A_i = \{1, \dots, d\} \setminus \{i\}$. In this case the reflected BSDE evolves in the closure of the time-dependent domain

$$\begin{aligned} D &= \{(t, y) = (t, y_1, \dots, y_d) \in \mathbb{R}^{d+1} : 0 \leq t \leq T, \\ (1.4) \quad y_i &\geq \max_{j \in A_i} (y_j - c_{ij}(t, X_t)), \text{ for all } i \in \{1, \dots, d\}\}. \end{aligned}$$

On the boundary of D a reflection occurs and in [HT] the authors refer to this as an oblique reflection. While this oblique reflection seems to have no clear relation to what is referred to as an oblique reflection in the context of (1.1), (1.2), it is still fair to refer to the problem in (1.3) as an obliquely reflected BSDE. However, we emphasize that the problems in (1.1), (1.2) and (1.3) are significantly different.

In this paper we consider the problem in (1.1), (1.2) in time-dependent domains and with underlying stochastic processes beyond Brownian motion. In light of (1.3), (1.4), and corresponding developments for reflected SDEs, see [C], [CGK], [LS], [NO], [S], and [T], it is natural to allow for time-dependent domains and in many cases this extra feature calls for additional arguments in comparison with the case of time-independent domains. In particular, we here consider (1.1), (1.2) in the context of time-dependent domains, and along the lines of [GP]. In addition, we allow the BSDE to be driven by a Wiener-Poisson type process and our main result is a generalization of [GP] and [O] to a time-dependent setting. In general, it seems difficult to generalize [GP] and its proofs beyond the assumption of convexity of the time-slices of the domain. Indeed, the assumption on convexity is heavily explored in [GP] and [O]. Beyond ensuring the existence of projections, convexity establishes the positivity of certain terms appearing when applying the Ito formula. In this sense, one may say that the arguments are slightly rigid as the structural assumption of convexity seems crucial. In our analysis, it turns out that we are only able to pull the arguments of [GP] through in the context of time-dependent domain having a similar rigidity in time. More precisely, in our case the time slices must be non-increasing and hence the domain must be non-expanding as a function of time. Under such a structural assumption though, we are able to generalize [GP] and [O] to a time-dependent setting. Finally, we note that it is an interesting

open problem to understand if, in analogy with the connection between optimal switching problems and the problem in (1.3), the problem in (1.1), (1.2) can be naturally associated to some stochastic optimization problem.

2. STATEMENT OF MAIN RESULT

In this section we state our main result. To do this properly we first briefly discuss the geometry and processes of Wiener-Poisson type and define the reflected BSDE studied in this paper.

2.1. Geometry. Given $d \geq 1$, we let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbb{R}^d and $|z| = \langle z, z \rangle^{1/2}$ be the Euclidean norm of $z \in \mathbb{R}^d$. Whenever $z \in \mathbb{R}^d$ and $r > 0$, we let $B_r(z)$ and $S_r(z)$ denote the ball and sphere of radius r , centered at z , respectively, i.e. $B_r(z) = \{y \in \mathbb{R}^d : |z - y| < r\}$ and $S_r(z) = \{y \in \mathbb{R}^d : |z - y| = r\}$. Moreover, given $F \subset \mathbb{R}^d$, $E \subset \mathbb{R}^d$, we let \bar{F} , \bar{E} be the closure of F and E , respectively, and we let $d(y, E)$ denote the Euclidean distance from $y \in \mathbb{R}^d$ to E . Given $d \geq 1$, $T > 0$ and an open, connected set $D' \subset \mathbb{R}^{d+1}$ we will refer to

$$D = D' \cap ([0, T] \times \mathbb{R}^d),$$

as a time-dependent domain.

Given D and $t \in [0, T]$, we define the time sections of D as

$$(2.1) \quad D_t = \{z : (t, z) \in D\}.$$

We assume that

$$(2.2) \quad D_t \neq \emptyset, D_t \text{ is open, bounded and connected for every } t \in [0, T],$$

and that

$$(2.3) \quad D_t \text{ is convex for every } t \in [0, T].$$

Furthermore, following [CGK], we let

$$l(r) = \sup_{\substack{s, t \in [0, T] \\ |s - t| \leq r}} \sup_{z \in \bar{D}_s} d(z, D_t),$$

be the modulus of continuity of the variation of D in time and we assume that

$$(2.4) \quad \lim_{r \rightarrow 0^+} l(r) = 0.$$

We also assume that

$$(2.5) \quad D_{t'} \subseteq D_t \text{ whenever } t' \geq t, t', t \in [0, T].$$

Note that (2.5) implies that

$$l(r) = \sup_{t \in [0, T], [t-r, t+r] \in [0, T]} \sup_{z \in \bar{D}_{t-r}} d(z, D_{t+r}).$$

We let ∂D and ∂D_t , for $t \in [0, T]$, denote the boundaries of D and D_t , respectively, and we let $N_t(z)$ denote the cone of inward normal vectors at $z \in \partial D_t$, $t \in [0, T]$. Note that it follows from (2.3) that $N_t(z) \neq \emptyset$ for every $z \in \partial D_t$, $t \in [0, T]$. In general, the cone $N_t(z)$ of inward normal vectors at $z \in \partial D_t$, $t \in [0, T]$, is defined as being equal to the set consisting of the union of the set $\{0\}$ and the set

$$\{v \in \mathbb{R}^d : v \neq 0, \exists \rho > 0 \text{ such that } B_\rho(z - \rho v / |v|) \subset \mathbb{R}^d \setminus D_t\}.$$

Note that this definition does not rule out the possibility of several unit inward normal vectors at the same boundary point. Given $N_t(z)$, we let $N_t^1(z) := N_t(z) \cap S_1(0)$, so that $N_t^1(z)$ contains the set of vectors in $N_t(z)$ with unit length. In this paper we consider reflected BSDEs in the setting of time-dependent domains D satisfying (2.2)-(2.5). Furthermore, reflection at $z \in \partial D_t$, $t \in [0, T]$, is considered in the direction of a unit spatial inward normal in the cone $N_t(z)$.

2.2. Processes of Wiener-Poisson type. Throughout the paper we let

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$$

be a complete Wiener-Poisson space in $\mathbb{R}^n \times \mathbb{R}^m \setminus \{0\}$ with Levy measure λ . In particular, $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space and $\{\mathcal{F}_t\}_{t \in [0, T]}$ is an increasing, right continuous family of complete sub σ -algebras of \mathcal{F} . We let $(W_t, \{F_t\})_{t \in [0, T]}$ be a standard Wiener process in \mathbb{R}^n and $(\mu_t, \{\mathcal{F}_t\})_{t \in [0, T]}$ be a martingale measure in $\mathbb{R}^m \setminus \{0\}$, which is assumed to be independent of W , and which corresponds to a standard Poisson random measure $p(t, A)$. Indeed, for any Borel measurable subset A of $\mathbb{R}^m \setminus \{0\}$ such that the Levy measure λ satisfies $\lambda(A) < +\infty$, we have

$$\mu_t(A) = p(t, A) - t\lambda(A)$$

where $p(t, A)$ satisfies

$$E[p(t, A)] = t\lambda(A).$$

We let $U := \mathbb{R}^m \setminus \{0\}$ and we let \mathcal{U} be its Borel σ -algebra. We assume that $\{\mathcal{F}_t\}_{t \in [0, T]}$ is the filtration generated by W_t and the jump process corresponding to the Poisson random measure p , augmented with the \mathbb{P} -null sets of \mathcal{F} , i.e.,

$$\mathcal{F}_t = \sigma \left(\int_{A \times [0, s]} p(ds, dx) : s \leq t, A \in \mathcal{U} \right) \vee \sigma(W_s, s \leq t) \vee \mathcal{F}_0,$$

where \mathcal{F}_0 denotes the \mathcal{P} -null sets of \mathcal{F} and $\sigma_1 \vee \sigma_2$ denotes, given two σ -algebras σ_1 and σ_2 , the σ -algebra generated by $\sigma_1 \cup \sigma_2$.

2.3. Reflected BSDEs. Given $T > 0$, we let $\mathcal{D}([0, T], \mathbb{R}^d)$ denote the set of càdlàg functions $v(t) = v_t : [0, T] \rightarrow \mathbb{R}^d$, i.e., functions which are right continuous and have left limits. We denote the set of functions $w(t) = w_t : [0, T] \rightarrow \mathbb{R}^d$ with bounded variation by $\mathcal{BV}([0, T], \mathbb{R}^d)$ and we let $|w|$ denote the total variation of $w \in \mathcal{BV}([0, T], \mathbb{R}^d)$. Recall that the total variation process $|w|$ is defined as

$$|w|_t = \sup \sum_{k=1}^n |w_{t_i} - w_{t_{i-1}}|, \quad 0 \leq t \leq T,$$

where the supremum is taken over all finite partitions $0 = t_0 < t_1 < \dots < t_n = t$. Furthermore, we have that

$$(2.6) \quad w_t = \int_0^t \nu_s d|w|_s$$

where ν_s is a vector of length 1, i.e., $|\nu_s| = 1$ for $|w|$ -almost all s . Let

$$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}, W_t, \mu_t, t \in [0, T])$$

be the complete Wiener-Poisson space in $\mathbb{R}^n \times \mathbb{R}^m \setminus \{0\}$, with Levy measure λ , as outlined above. Let $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ be the space of square integrable, \mathcal{F}_T -adapted random variables and let $L^2(U, \mathcal{U}, \lambda; \mathbb{R}^d)$ be the space of functions which are \mathcal{U} -measurable, maps values in U to \mathbb{R}^d , and which are square integrable on U with respect to the Levy-measure λ . In the following we let the norm

$$\|z\| := \left(\sum_{i,j} |z_{ij}|^2 \right)^{1/2}$$

be defined on real-valued $(d \times n)$ -dimensional matrices and we define the norm

$$\|u(e)\| := \left(\int_V |u(e)|^2 \lambda(de) \right)^{1/2}$$

on $L^2(U, \mathcal{U}, \lambda; \mathbb{R}^d)$. Let $\xi = (\xi_1, \dots, \xi_d)$ be such that

$$(2.7) \quad \xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \text{ and } \xi \in D_T \text{ a.s.}$$

Let $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R}^{d \times n} \times L^2(U, \mathcal{U}, \lambda; \mathbb{R}^d) \rightarrow \mathbb{R}^d$ be a function such that

$$\begin{aligned}
 (i) \quad & (\omega, t) \rightarrow f(\omega, t, y, z, u) \text{ is } \mathcal{F}_t \text{ progressively measurable whenever} \\
 & (y, z, u) \in \mathbb{R}^d \times \mathbb{R}^{d \times n} \times L^2(U, \mathcal{U}, \lambda; \mathbb{R}^d), \\
 (ii) \quad & E \left[\int_0^T |f(\omega, t, 0, 0, 0)|^2 dt \right] < \infty, \\
 (iii) \quad & |f(\omega, t, y, z, u) - f(\omega, t, y', z', u')| \leq c(|y - y'| + \|z - z'\| + \|u - u'\|) \\
 & \text{for some constant } c \text{ whenever} \\
 (2.8) \quad & (y, z, u), (y', z', u') \in \mathbb{R}^d \times \mathbb{R}^{d \times n} \times L^2(U, \mathcal{U}, \lambda; \mathbb{R}^d), (\omega, t) \in \Omega \times [0, T].
 \end{aligned}$$

In the context of BSDEs, ξ and f are usually referred to as terminal value and driver of the BSDE, respectively. We are now ready to formulate the notion of reflected BSDE considered in this paper.

Definition 1. Let $d \geq 1$ and $T > 0$. Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (2.2). Given (ξ, f) as in (2.7)-(2.8), a quadruple $(Y_t, Z_t, U_t, \Lambda_t)$ of progressively measurable processes with values in $\mathbb{R}^d \times \mathbb{R}^{d \times m} \times L^2(U, \mathcal{U}, \lambda; \mathbb{R}^d) \times \mathbb{R}^d$ is said to be a solution to a reflected BSDE, with reflection in the inward spatial normal direction, in D , and with data (ξ, f) , if the following holds. $Y \in \mathcal{D}([0, T], \mathbb{R}^d)$, Z and U are predictable processes, and

$$\begin{aligned}
 (i) \quad & E \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] < \infty, \\
 (ii) \quad & E \left[\int_0^T \|Z_t\|^2 dt + \int_0^T \int_U |U_s(e)|^2 \lambda(de) ds \right] < \infty, \\
 (iii) \quad & Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds + \Lambda_T - \Lambda_t \\
 & \quad - \int_t^T Z_s dW_s - \int_t^T \int_U U_s(e) \mu(de, ds) \quad \text{a.s.}, \\
 (iv) \quad & Y_t \in \overline{D_t} \quad \text{a.s.},
 \end{aligned}$$

whenever $t \in [0, T]$. Furthermore, $\Lambda \in \mathcal{BV}([0, T], \mathbb{R}^d)$ and

$$\begin{aligned}
 (v) \quad & \Lambda_t = \int_0^t \gamma_s d|\Lambda|_s, \quad \gamma_s \in N_s^1(Y_s) \text{ whenever } Y_s \in \partial D_s, \\
 (vi) \quad & d|\Lambda|(\{t \in [0, T] : (t, Y_t) \in D\}) = 0.
 \end{aligned}$$

2.4. Statement of the main result. Concerning reflected BSDEs we establish the following result.

Theorem 2.1. Let $d \geq 1$ and $T > 0$. Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (2.2)-(2.5) and assume the terminal data ξ and driver f satisfy (2.7)-(2.8). Then there exists a unique solution $(Y_t, Z_t, U_t, \Lambda_t)$ to the reflected BSDE, with reflection in the inward spatial normal direction, in D , and with data (ξ, f) , in the sense of Definition 1.

2.5. Organization of the paper. The rest of the paper is organized as follows. Section 3 is of preliminary nature and we here focus on the geometry of the time-dependent domain as well as smooth approximations of it. We also recall the Ito formula in the context of Wiener-Poisson processes. In section 4 we introduce a sequence of non-reflected BSDEs, constructed by penalization techniques, and develop a number of technical lemmas for these. Finally, using the results of section 4, we prove the main result in section 5.

3. PRELIMINARIES

Let $d \geq 1$ and $T > 0$. Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (2.2) - (2.5). Let $N = N_t(z) = N(t, z)$ denote the cone of inward normal vectors given for all $z \in \partial D_t$, $t \in [0, T]$. Note that by (2.3) there exists, for any $y \in \mathbb{R}^d \setminus \overline{D}_t$, $t \in [0, T]$, at least one projection of y onto ∂D_t along N_t , denoted $\pi(t, y)$, which satisfies

$$|y - \pi(t, y)| = d(y, D_t).$$

To have $\pi(t, \cdot)$ well-defined for all $y \in \mathbb{R}^d$ we also let $\pi(t, y) = y$ whenever $y \in \overline{D}_t$. The following lemma summarizes a few standard results from convex analysis.

Lemma 3.1. *Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (2.2) and assume (2.3) and (2.5). Then the following holds whenever $t \in [0, T]$:*

- (i) $\langle y' - y, y - \pi(t, y) \rangle \leq 0$, for $(y, y') \in \mathbb{R}^d \times \overline{D}_t$, and
- (ii) $\langle y' - y, y - \pi(t, y) \rangle \leq \langle y' - \pi(t, y'), y - \pi(t, y) \rangle$,
- (iii) $|\pi(t, y) - \pi(t, y')| \leq |y - y'|$,

whenever $(y, y') \in \mathbb{R}^d \times \mathbb{R}^d$. Furthermore, there exists $P_T \in D_T$ and γ , $1 \leq \gamma < \infty$, depending on $d(P_T, \partial D_T)$, such that

$$(iv) \quad \langle y - P_T, y - \pi(t, y) \rangle \geq \gamma^{-1} |y - \pi(t, y)|, \text{ for any } y \in \mathbb{R}^d, t \in [0, T].$$

3.1. Geometry of time-dependent domains - smooth approximations. Note that the assumptions in (2.2), (2.3), and (2.5) contain no particular smoothness assumption. Instead, we will in the following construct smooth approximations of D to enable the use of Ito's formula. In the following we let $h(t, y) = d(y, D_t)$ whenever $y \in \mathbb{R}^d$, $t \in [0, T]$, with the convention that $h(t, y) = 0$ if $y \in \overline{D}_t$. Assuming that $D_T \neq \emptyset \neq D_0$ we let $h(t, y) = h(T, y)$ whenever $y \in \mathbb{R}^d$, $t > T$, and $h(t, y) = h(0, y)$ whenever $y \in \mathbb{R}^d$, $t < 0$. Using this notation we see that

$$\overline{D} = \{(t, x) \in [0, T] \times \mathbb{R}^d \mid h(t, x) = 0\}.$$

Let $\phi = \phi(s, y)$ be a smooth mollifier in \mathbb{R}^{d+1} , i.e., $\phi \in C_0^\infty(\mathbb{R}^{d+1})$, $0 \leq \phi \leq 1$, the support of ϕ is contained in the Euclidean unit ball in \mathbb{R}^{d+1} , centered at 0, and $\int \phi dy ds = 1$. Let, for $\delta > 0$ small, $\phi_\delta(s, y) = \delta^{n+1} \phi(\delta^{-1}s, \delta^{-1}y)$. Based on ϕ_δ we let, whenever $(t, y) \in \mathbb{R} \times \mathbb{R}^d$,

$$h_\delta(t, y) = (\phi_\delta * h)(t, y) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \phi_\delta(t - s, y - x) h(s, x) dx ds$$

be a smooth mollification of h . Furthermore, we let

$$h(F, G) = \max(\sup\{d(y, F) : y \in G\}, \sup\{d(y, G) : y \in F\})$$

denote the Hausdorff distance between the sets $F, G \subset \mathbb{R}^d$. Based on h_δ we introduce a smooth approximation of D as follows. Given η fixed and $\delta > 0$, we let

$$D_\delta^\eta = \{(t, x) \in [0, T] \times \mathbb{R}^d \mid h_\delta(t, x) < \eta\}.$$

Then D_δ^η converges to $D^\eta := \{(t, x) \in [0, T] \times \mathbb{R}^d \mid h(t, x) < \eta\}$ in the Hausdorff distance sense as $\delta \rightarrow 0$. Note that D_δ^η is a C^∞ -smooth domain. Hence, letting $\delta, \eta \rightarrow 0$ we have the following lemma.

Lemma 3.2. *Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (2.2) - (2.5). Then, for any $\epsilon > 0$ there exists a time-dependent domain $D_\epsilon \subset \mathbb{R}^{d+1}$ satisfying (2.2) - (2.5) such that D_ϵ is C^∞ -smooth and*

$$h(D_t, D_{\epsilon, t}) < \epsilon \text{ for all } t \in [0, T],$$

where D_t is as defined in (2.1), and $D_{\epsilon, t} = \{x : (x, t) \in D_\epsilon\}$.

Let, for all $t \in [0, T]$, $N_\epsilon = N_{\epsilon,t}(z) = N_\epsilon(t, z)$ denote the cone of inward normal vectors at $z \in \partial D_{\epsilon,t}$. Due to the smoothness of $\partial D_{\epsilon,t}$, $N_\epsilon(t, z)$ consists of a single vector. For any $y \in \mathbb{R}^d \setminus \overline{D_{\epsilon,t}}$, $t \in [0, T]$, we let $\pi_\epsilon(t, y)$ denote the projection of y onto $\partial D_{\epsilon,t}$ along this unique direction. To have $\pi_\epsilon(t, \cdot)$ well-defined for all $y \in \mathbb{R}^d$ we also let $\pi_\epsilon(t, y) = y$ whenever $y \in \overline{D_{\epsilon,t}}$. In this setting, the following lemma can be proven as Lemma 2.2 in [GP] as we are only considering fixed time slices D_t of D .

Lemma 3.3. *Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (2.2) - (2.5) and let D_ϵ be constructed as above. There exists a constant c such that, if $\epsilon \in (0, 1)$, $y \in \mathbb{R}^d$ and $t \in [0, T]$, then*

$$\begin{aligned} (i) \quad & |\pi(t, y) - \pi_\epsilon(t, y)| \leq c\sqrt{\epsilon^2 + \epsilon d(y, D_{\epsilon,t})}, \\ (ii) \quad & |\pi(t, y) - \pi_\epsilon(t, y)| \leq c\sqrt{\epsilon^2 + \epsilon d(y, D_t)}. \end{aligned}$$

The following lemma is a corollary of Lemma 3.3.

Lemma 3.4. *Let D and D_ϵ be as in the statement of Lemma 3.3. There exists a constant c such that, if $\epsilon \in (0, 1)$ and $y \in \mathbb{R}^d$, $t \in [0, T]$, then*

$$\begin{aligned} (i) \quad & |\pi(t, y) - \pi_\epsilon(t, y)| \leq c\sqrt{\epsilon}(1 + d(y, D_{\epsilon,t})), \\ (ii) \quad & |\pi(t, y) - \pi_\epsilon(t, y)| \leq c\sqrt{\epsilon}\sqrt{d(y, D_{t,\epsilon})} \text{ whenever } d(y, D_{\epsilon,t}) > \epsilon. \end{aligned}$$

We here also recall Ito's formula in the context of Wiener-Poisson processes, see [OS]. Here and in the following, we denote by $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$ the space of functions $\varphi(t, y) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ which are once continuously differentiable with respect to $t \in [0, T]$ and twice continuously differentiable with respect to $y \in \mathbb{R}^d$ and we let A^* denote the transpose of the matrix A .

Lemma 3.5. *Let Y_t be a Levy process such that*

$$dY_t = f_t dt + \sigma_t dW_t + \int_U U_t(e) \mu(de, dt)$$

and let $\varphi(t, y) \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^d, \mathbb{R})$. Then

$$\begin{aligned} d\varphi(t, Y_t) &= \partial_t \varphi(t, Y_t) dt + (\nabla \varphi(t, Y_{t-})) \left[f_t dt + \sigma_t dW_t + \int_U U_s(e) \mu(de, ds) \right] \\ &+ \sum_{i,j} \frac{1}{2} (\sigma_t \sigma_t^*)_{ij} \partial_{y_i y_j}^2 \varphi_\epsilon(t, Y_t) dt \\ &+ \int_U [\varphi(t, Y_{t-} + U_t(e)) - \varphi(t, Y_{t-}) - \langle \nabla \varphi(t, Y_{t-}), U_t(e) \rangle] p(de, dt). \end{aligned}$$

Based on the smooth domain D_ϵ we define the function $\varphi_\epsilon(t, y) := (d(y, D_{\epsilon,t}))^2 = |y - \pi_\epsilon(t, y)|^2$ to which Ito's formula needs to be applied in the proof of Theorem 2.1,. Note that although D_ϵ is a smooth domain, the second (spatial) derivative of φ_ϵ is not continuous at the boundary of $D_{\epsilon,t}$ and thus Lemma 3.5 is not directly applicable. To enable the use of Ito's formula we therefore proceed along the lines of [LS], see also [GP], and extend our distance function φ_ϵ across the boundary and into the domain $D_{\epsilon,t}$. Indeed, since $\partial D_{\epsilon,t}$ is smooth there exists a neighbourhood $V_{\epsilon,t}$ of $\partial D_{\epsilon,t}$ such that, for $y \in D_{\epsilon,t} \cap V_{\epsilon,t}$, there exists a unique pair $(x, s) \in \partial D_{\epsilon,t} \times \mathbb{R}^+$ such that $y = x + s\gamma$, where $\gamma \in N_{\epsilon,t}^1(x)$. Recall that $N_{\epsilon,t}^1(x)$, the cone of unit inward normal vectors to $D_{\epsilon,t}$, at $x \in \partial D_{\epsilon,t}$, contains only a single vector. By the convexity of $D_{\epsilon,t}$ we also have

$$y = x + s\gamma, \text{ for } x = \pi_\epsilon(t, y), \ s = -d(y, D_{\epsilon,t}), \ \gamma \in N_{\epsilon,t}^1(\pi_\epsilon(t, x)),$$

whenever $y \in \mathbb{R}^d \setminus \overline{D_{\epsilon,t}}$. Hence, for $t \in [0, T]$ fixed, we can define a smooth map $\phi_\epsilon : \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$\begin{aligned}\phi_\epsilon(t, y) &= s & \text{when } y \in (\mathbb{R}^d \setminus \overline{D_{\epsilon,t}}) \cup (V_{\epsilon,t} \cap \overline{D_{\epsilon,t}}), \\ \phi_\epsilon(t, y) &> 0 & \text{otherwise.}\end{aligned}$$

Using such a function ϕ_ϵ we have

$$\begin{aligned}D_\epsilon &= \{(t, y) : t \in [0, T], y \in \mathbb{R}^d, \phi_\epsilon(t, y) > 0\}, \\ \partial D_{\epsilon,t} &= \{y \in \mathbb{R}^d, \phi_\epsilon(t, y) = 0\}, \text{ for } t \in [0, T], \\ (\mathbb{R}^d \times [0, T]) \setminus \overline{D_\epsilon} &= \{(t, y) : t \in [0, T], y \in \mathbb{R}^d, \phi_\epsilon(t, y) < 0\}.\end{aligned}$$

Note that $\phi_\epsilon(t, y)$ is smooth also across the boundary of $D_{\epsilon,t}$ and that $\varphi_\epsilon(t, y) = (\phi_\epsilon(t, y)^-)^2$. Following [GP], we can now take an approximating sequence of smooth functions $\{g_n\}_{n \geq 0}$, tending to $g(x) = (x^-)^2$ as $n \rightarrow \infty$, and construct a sequence of smooth functions $\{\varphi_\epsilon^n(t, y) = g_n(\phi_\epsilon(t, y))\}_{n \geq 0}$ such that Ito's formula can be applied to φ_ϵ^n for every $n \geq 0$ and such that $\varphi_\epsilon^n(t, y)$, $\frac{\partial}{\partial t}\varphi_\epsilon^n(t, y)$, $\frac{\partial}{\partial y_i}\varphi_\epsilon^n(t, y)$, $\frac{\partial^2}{\partial y_i \partial y_j}\varphi_\epsilon^n(t, y)$ tend to $\varphi_\epsilon(t, y)$, $\frac{\partial}{\partial t}\varphi_\epsilon(t, y)$, $\frac{\partial}{\partial y_i}\varphi_\epsilon(t, y)$, $\frac{\partial^2}{\partial y_i \partial y_j}\varphi_\epsilon(t, y)$, respectively, as $n \rightarrow \infty$. Having such an approximation in mind, we will from here on in slightly abuse notation and apply the Ito formula directly to $\varphi_\epsilon(t, y)$.

Finally, the following lemma is the result of the geometric assumptions on D that we will use in the context of Ito's formula.

Lemma 3.6. *Let D and D_ϵ be as in the statement of Lemma 3.3 and let $\varphi_\epsilon(t, y)$ be defined as*

$$\varphi_\epsilon(t, y) := (d(y, D_{\epsilon,t}))^2 = |y - \pi_\epsilon(t, y)|^2, \quad t \in [0, T], \quad y \in \mathbb{R}^d.$$

Then

$$(3.1) \quad \begin{aligned}(i) \quad &\partial_t \varphi_\epsilon(t, y) \geq 0, \\ (ii) \quad &\partial_{y_i y_j}^2 \varphi_\epsilon(t, y) \xi_i \xi_j \geq 0,\end{aligned}$$

whenever $t \in [0, T]$, $y, \xi \in \mathbb{R}^d$, and

$$(3.2) \quad \varphi_\epsilon(t, y + z) - \varphi_\epsilon(t, y) - \langle \nabla \varphi_\epsilon(t, y), z \rangle \geq 0$$

whenever $t \in [0, T]$, $y, z \in \mathbb{R}^d$.

Proof. (3.1) (i) follows from (2.5) and (3.1) (ii) follows from the convexity of $D_{\epsilon,t}$. Finally, Taylors formula and (3.1) (ii) yields (3.2). \square

4. ESTIMATES FOR APPROXIMATING PROBLEMS: TECHNICAL LEMMAS

To prove the existence part of Theorem 2.1 we use the method of penalization. Indeed, for each $n \geq 1$, we construct a quadruple $(Y_t^n, Z_t^n, U_t^n, \Lambda_t^n)$ through penalization and we prove that $(Y_t^n, Z_t^n, U_t^n, \Lambda_t^n)$ converges, as $n \rightarrow \infty$, to a solution $(Y_t, Z_t, U_t, \Lambda_t)$ of the reflected backward stochastic differential equation, with reflection in the inward spatial normal direction, in D , and with data (ξ, f) , as defined in Definition 1. Uniqueness is then proved by an argument based on Ito's formula. In the following we let (ξ, f) be as in (2.7)-(2.8), and we let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (2.2) - (2.5). Furthermore, we let c denote a generic constant which may change value from line to line.

4.1. Construction of $(Y_t^n, Z_t^n, U_t^n, \Lambda_t^n)$. Let, for any $n \in \mathbb{Z}_+$,

$$(4.1) \quad f_n(t, y, z, u) := f(t, y, z, u) - n(y - \pi(t, y)).$$

Then, for n fixed, f_n satisfies (2.8) since π has the Lipschitz property in space, see Lemma 3.1 (iii). Hence, using results concerning existence and uniqueness for (unconstrained) BSDEs driven by Wiener-Poisson type processes, see Lemma 2.4 in [TL], we can conclude that there exist, for each $n \in \mathbb{Z}_+$, a unique triple (Y_t^n, Z_t^n, U_t^n) and a constant c_n , $1 \leq c_n < \infty$, such that

$$(4.2) \quad \begin{aligned} (i) \quad & E \left[\sup_{0 \leq t \leq T} |Y_t^n|^2 \right] \leq c_n, \\ (ii) \quad & E \left[\int_0^T \|Z_t^n\|^2 dt + \int_0^T \int_U |U_s^n(e)|^2 \lambda(de) ds \right] < \infty, \\ (iii) \quad & Y_t^n = \xi + \int_t^T f_n(s, Y_s^n, Z_s^n, U_s^n) ds \\ & - \int_t^T Z_s^n dW_s - \int_t^T \int_U U_s^n(e) \mu(de, ds). \end{aligned}$$

Note also that from [TL] we have $Y^n \in \mathcal{D}([0, T], \mathbb{R}^d)$. Given (Y_t^n, Z_t^n, U_t^n) we define, for $n \in \mathbb{Z}_+$, the process Λ_t^n through

$$(4.3) \quad \Lambda_t^n = -n \int_0^t (Y_s^n - \pi(s, Y_s^n)) ds.$$

Note that

$$\Lambda_t^n = \int_0^t \frac{-(Y_s^n - \pi(s, Y_s^n))}{|Y_s^n - \pi(s, Y_s^n)|} d|\Lambda^n|_s$$

and that $-(Y_s^n - \pi(s, Y_s^n))/|Y_s^n - \pi(s, Y_s^n)|$ is an element in the inwards directed normal cone to D_s at $\pi(s, Y_s^n) \in \partial D_s$. Furthermore, using (4.1) and (4.3) we see that (4.2) (iii) can be rewritten as

$$\begin{aligned} Y_t^n &= \xi + \int_t^T f(s, Y_s^n, Z_s^n, U_s^n) ds + \Lambda_T^n - \Lambda_t^n \\ &\quad - \int_t^T Z_s^n dW_s - \int_t^T \int_U U_s^n(e) \mu(de, ds), \end{aligned}$$

for all $t \in [0, T]$. Recall that Y^n, Λ^n, U^n, W_t and Z^n are multi-dimensional processes. In particular, $Y_t^n = (Y_t^{1,n}, \dots, Y_t^{d,n})$, $\Lambda_t^n = (\Lambda_t^{1,n}, \dots, \Lambda_t^{d,n})$, $U_t^n = (U_t^{1,n}, \dots, U_t^{d,n})$, $W_t = (W_t^1, \dots, W_t^n)$ and Z_t^n is a $d \times n$ -matrix with entries $Z_t^{i,j,n}$ and columns $Z_t^{j,n}$.

4.2. A priori estimates for $(Y_t^n, Z_t^n, U_t^n, \Lambda_t^n)$.

Lemma 4.1. *There exists a constant c , $1 \leq c < \infty$, independent of n , such that*

$$\begin{aligned} (i) \quad & E \left[\sup_{0 \leq t \leq T} |Y_t^n|^2 \right] \leq c, \\ (ii) \quad & E \left[\int_t^T \|Z_s^n\|^2 ds + \int_t^T \int_U |U_s^n(e)|^2 \lambda(de) ds \right] \leq c, \quad t \in [0, T], \\ (iii) \quad & E \left[n \int_t^T |Y_s^n - \pi(s, Y_s^n)| ds \right] \leq c, \quad t \in [0, T]. \end{aligned}$$

Proof. Let $P_T \in D_T$ be as in Lemma 3.1 (iv). Applying Ito's formula to the process $|Y_t^n - P_T|^2$ we deduce, for $t \in [0, T]$, that

$$\begin{aligned}
 & |Y_t^n - P_T|^2 + \int_t^T \|Z_s^n\|^2 ds + \int_t^T \int_U |U_s^n(e)|^2 p(de, ds) \\
 = & |\xi - P_T|^2 + 2 \int_t^T \langle Y_s^n - P_T, f(s, Y_s^n, Z_s^n, U_s^n) \rangle ds - 2 \int_t^T \langle Y_s^n - P_T, Z_s^n dW_s \rangle \\
 (4.4) \quad & - 2 \int_t^T \int_U \langle Y_s^n - P_T, U_s^n(e) \rangle \mu(de, ds) - 2n \int_t^T \langle Y_s^n - P_T, Y_s^n - \pi(s, Y_s^n) \rangle ds.
 \end{aligned}$$

Let

$$A_n := |Y_t^n - P_T|^2 + \int_t^T \|Z_s^n\|^2 ds + \int_t^T \int_U |U_s^n(e)|^2 p(de, ds).$$

Rearranging (4.4), we find that

$$\begin{aligned}
 & A_n + 2n \int_t^T \langle Y_s^n - P_T, Y_s^n - \pi(s, Y_s^n) \rangle ds \\
 = & |\xi - P_T|^2 + 2 \int_t^T \langle Y_s^n - P_T, f(s, Y_s^n, Z_s^n, U_s^n) \rangle ds \\
 & - 2 \int_t^T \langle Y_s^n - P_T, Z_s^n dW_s \rangle - 2 \int_t^T \int_U \langle Y_s^n - P_T, U_s^n(e) \rangle \mu(de, ds).
 \end{aligned}$$

Furthermore, using Lemma 3.1 (iv) we see that

$$\begin{aligned}
 & A_n + 2\gamma^{-1}n \int_t^T |Y_s^n - \pi(s, Y_s^n)| ds \\
 \leq & |\xi - P_T|^2 + 2 \int_t^T \langle Y_s^n - P_T, f(s, Y_s^n, Z_s^n, U_s^n) \rangle ds \\
 (4.5) \quad & - 2 \int_t^T \langle Y_s^n - P_T, Z_s^n dW_s \rangle + 2 \int_t^T \int_U \langle Y_s^n - P_T, U_s^n(e) \rangle \mu(de, ds).
 \end{aligned}$$

Next, taking expectations in (4.5) and using the fact that μ is a martingale measure, we can conclude that

$$(4.6) \quad E \left[A_n + 2\gamma^{-1}n \int_t^T |Y_s^n - \pi(s, Y_s^n)| ds \right] \leq I_{t,T}$$

where

$$(4.7) \quad I_{t,T} = E \left[|\xi - P_T|^2 \right] + 2E \left[\int_t^T \langle Y_s^n - P_T, f(s, Y_s^n, Z_s^n, U_s^n) \rangle ds \right].$$

Using the Lipschitz character of f , (2.8) (iii), and the inequality $ab \leq \eta a^2 + \frac{b^2}{4\eta}$ it follows that we can estimate $I_{t,T}$ as,

$$\begin{aligned}
 I_{t,T} & \leq c \left(1 + E \left[\int_t^T (|f(s, P_T, 0, 0)|^2 + (1 + 2\eta)|Y_s^n - P_T|^2 ds) \right] \right) \\
 (4.8) \quad & + cE \left[\int_t^T \frac{1}{\eta} \left(\|Z_s^n\|^2 + \int_U |U_s^n(e)|^2 \lambda(de) \right) ds \right]
 \end{aligned}$$

where $\eta > 0$ is a degree of freedom and $c, 1 \leq c < \infty$, is a constant depending on ξ, f and $\text{diam}(D_T)$. If we let η be such that $c/\eta \leq 1/2$, it follows from (4.6) and

(4.8), after recalling the definition of A_n , that

$$(4.9) \leq c \left(1 + E \left[\int_t^T |Y_s^n - P_T|^2 ds \right] \right).$$

Using (4.9) and Gronwall's lemma we deduce that

$$(4.10) \quad \sup_{0 \leq t \leq T} E[|Y_t^n - P_T|^2] \leq \tilde{c} \exp(\tilde{c}T)$$

where \tilde{c} is independent of n . In particular, we can conclude that there exists \hat{c} , $1 \leq \hat{c} < \infty$, independent of n , such that

$$(4.11) \quad (i) \quad \sup_{0 \leq t \leq T} E[|Y_t^n|^2] \leq \hat{c}, \quad \text{and} \\ (ii) \quad E \left[\int_0^T \|Z_s^n\|^2 ds + \int_0^T \int_U |U_s^n(e)|^2 \lambda(de) ds \right] \leq \hat{c}.$$

We next prove that there exists \check{c} , $1 \leq \check{c} < \infty$, independent of n , such that

$$(4.12) \quad E \left[\sup_{0 \leq t \leq T} |Y_t^n|^2 \right] \leq \check{c}.$$

To do this we first note, using (4.5) and the Lipschitz character of f , that, for a constant c , $1 \leq c < \infty$,

$$(4.13) \quad \begin{aligned} c^{-1} |Y_t^n - P_T|^2 &\leq |\xi - P_T|^2 + \left| \int_t^T \langle Y_s^n - P_T, f(s, P_T, 0, 0) \rangle ds \right| \\ &+ \left| \int_t^T |Y_s^n - P_T| (|Y_s^n - P_T| + \|Z_s^n\| + \|U_s^n\|_2) ds \right| \\ &+ \left| \int_t^T \langle Y_s^n - P_T, Z_s^n dW_s \rangle \right| + \left| \int_t^T \int_U \langle Y_s^n - P_T, U_s^n(e) \rangle \mu(de, ds) \right|. \end{aligned}$$

We treat the last two terms on the right hand side of (4.13) using Hölders inequality and the Burkholder-Davis-Gundy inequality. Indeed, applying these yields

$$(4.14) \quad \begin{aligned} &E \left[\sup_{0 \leq t \leq T} \left| \int_t^T \langle Y_s^n - P_T, Z_s^n dW_s \rangle \right| \right] \\ &\leq \left(E \left[\sup_{0 \leq t \leq T} |Y_t^n - P_T|^2 \right] \right)^{1/2} \left(E \left[\sup_{0 \leq t \leq T} \left| \int_t^T Z_s^n dW_s \right|^2 \right] \right)^{1/2} \\ &\leq c \left(E \left[\sup_{0 \leq t \leq T} |Y_t^n - P_T|^2 \right] \right)^{1/2} \left(E \left[\int_0^T \|Z_s^n\|^2 ds \right] \right)^{1/2}. \end{aligned}$$

Similarly,

$$\begin{aligned} &E \left[\sup_{0 \leq t \leq T} \left| \int_t^T \int_U \langle Y_s^n - P_T, U_s^n(e) \rangle \mu(de, ds) \right| \right] \\ &\leq c \left(E \left[\sup_{0 \leq t \leq T} |Y_t^n - P_T|^2 \right] \right)^{1/2} \left(E \left[\int_0^T \int_U |U_s^n(e)|^2 \lambda(de) ds \right] \right)^{1/2}. \end{aligned}$$

Using the above estimates as well as (4.10), (4.11) and assumption (2.2) we get after taking expectation in (4.13) that

$$E \left[\sup_{0 \leq t \leq T} |Y_t^n - P_T|^2 \right] \leq c \left(1 + \left(E \left[\sup_{0 \leq t \leq T} |Y_t^n - P_T|^2 \right] \right)^{1/2} \right)$$

from which we conclude that $E[\sup_{0 \leq t \leq T} |Y_t^n - P_T|^2] \leq c$ for some constant $1 \leq c < \infty$ and, consequently, that (4.12) holds. Finally, starting from (4.5) and repeating the arguments above we also deduce that

$$E \left[n \int_t^T |Y_s^n - \pi(s, Y_s^n)| ds \right] \leq \hat{c}$$

for some constant \hat{c} independent of n . This completes the proof of Lemma 4.1. \square

4.3. Uniform control of $d(Y_t^n, D_t)$. We here prove the following lemma.

Lemma 4.2. *Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (2.2) - (2.5). Let, for $\epsilon > 0$ small, D_ϵ be as in Lemma 3.2. Then there exist $\epsilon_0 > 0$ and c , $1 \leq c < \infty$, both independent of n , such that*

$$\begin{aligned} (i) \quad & E[\sup_{0 \leq t \leq T} d(Y_t^n, D_{\epsilon,t})^2] \leq c \left(\frac{1}{n} + \epsilon + n\epsilon^2 \right), \\ (ii) \quad & E \left[\int_0^T (d(Y_t^n, D_{\epsilon,t}))^2 dt \right] \leq c \left(\frac{1}{n^2} + \frac{\epsilon}{n} + \epsilon^2 \right), \end{aligned}$$

whenever $0 < \epsilon < \epsilon_0$ and $n \geq 1$.

Proof. Let $\varphi_\epsilon(t, Y_t) = d(Y_t^n, D_{\epsilon,t})^2 = |Y_t^n - \pi_\epsilon(t, Y_t^n)|^2$. Then $\nabla_y \varphi_\epsilon(t, Y_t) = 2(Y_t^n - \pi_\epsilon(t, Y_t^n))$. Using the Ito formula of Lemma 3.5 we see that

$$\begin{aligned} & \varphi_\epsilon(t, Y_t^n) + \int_t^T (\partial_s \varphi_\epsilon)(s, Y_s^n) ds + \frac{1}{2} \int_t^T \sum_{i,j} (Z_s^n Z_s^{n,*})_{ij} \partial_{y_i y_j}^2 \varphi_\epsilon(s, Y_s) ds \\ & + \int_t^T \int_U [\varphi_\epsilon(s, Y_{s-}^n + U_s^n(e)) - \varphi_\epsilon(Y_{s-}^n) - \langle \nabla \varphi_\epsilon(s, Y_{s-}^n), U_s^n(e) \rangle] p(de, ds) \\ (4.15) \quad & = \varphi_\epsilon(T, \xi) + I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &= 2 \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), f(s, Y_s, Z_s^n, U_s^n) \rangle ds, \\ I_2 &= -2 \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), Z_s^n dW_s \rangle, \\ I_3 &= -2n \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), Y_s^n - \pi(s, Y_s^n) \rangle ds, \\ (4.16) \quad I_4 &= -2 \int_t^T \int_U \langle Y_s^n - \pi_\epsilon(s, Y_s^n), U_s^n(e) \rangle \mu(de, ds). \end{aligned}$$

Using Lemma 3.6 we see that

$$\begin{aligned} & \int_t^T \partial_s \varphi_\epsilon(s, Y_s^n) ds + \frac{1}{2} \int_t^T \sum_{i,j} (Z_s^n Z_s^{n,*})_{ij} \partial_{y_i y_j}^2 \varphi_\epsilon(s, Y_s) ds \\ & + \int_t^T \int_U [\varphi_\epsilon(s, Y_{s-}^n + U_s^n(e)) - \varphi_\epsilon(Y_{s-}^n) - \langle \nabla \varphi_\epsilon(s, Y_{s-}^n), U_s^n(e) \rangle] p(de, ds) \geq 0, \end{aligned}$$

and hence

$$(4.17) \quad \varphi_\epsilon(t, Y_t^n) \leq \varphi_\epsilon(T, \xi) + I_1 + I_2 + I_3 + I_4.$$

Since $\xi \in D_T$ a.s. we see, using Lemma 3.3, that $\varphi_\epsilon(T, \xi) \leq c\epsilon^2$ a.s. To simplify the notation in what follows, we define $\chi_\epsilon(t, y) : [0, T] \times \mathbb{R}^d \rightarrow \{0, 1\}$ as

$$\chi_\epsilon(t, y) = \begin{cases} 1 & \text{if } d(y, D_{\epsilon,t}) > \epsilon \\ 0 & \text{otherwise} \end{cases}.$$

We first focus on the term I_1 in (4.16). Then, using the above introduced notation we see that

$$\begin{aligned} I_1 &\leq 2 \int_t^T |\varphi_\epsilon(s, Y_s^n)|^{1/2} |f(s, Y_s^n, Z_s^n, U_s^n)| \chi_\epsilon(s, Y_s^n) ds \\ &\quad + 2 \int_t^T |\varphi_\epsilon(s, Y_s^n)|^{1/2} |f(s, Y_s^n, Z_s^n, U_s^n)| (1 - \chi_\epsilon(s, Y_s^n)) ds. \end{aligned}$$

Furthermore, by the inequality $ab \leq \eta a^2 + \frac{b^2}{4\eta}$ and $x \leq \max\{1, x^2\}$,

$$\begin{aligned} (i) \quad & 2|\varphi_\epsilon(s, Y_s^n)|^{1/2} |f(s, Y_s^n, Z_s^n, U_s^n)| \chi_\epsilon(s, Y_s^n) \\ & \leq \frac{n}{4} \varphi_\epsilon(s, Y_s^n) \chi_\epsilon(s, Y_s^n) + \frac{4}{n} |f(s, Y_s^n, Z_s^n, U_s^n)|^2 \chi_\epsilon(s, Y_s^n), \\ (ii) \quad & 2|\varphi_\epsilon(s, Y_s^n)|^{1/2} |f(s, Y_s^n, Z_s^n, U_s^n)| (1 - \chi_\epsilon(s, Y_s^n)) \\ & \leq 2(\epsilon + \epsilon |f(s, Y_s^n, Z_s^n, U_s^n)|^2) (1 - \chi_\epsilon(s, Y_s^n)). \end{aligned}$$

Next, focusing on the term I_3 , we have by the bilinearity of $\langle \cdot, \cdot \rangle$ that

$$\begin{aligned} I_3 &= -2n \int_t^T |Y_s^n - \pi_\epsilon(s, Y_s^n)|^2 \chi_\epsilon(s, Y_s^n) ds \\ &\quad - 2n \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), \pi_\epsilon(s, Y_s^n) - \pi(s, Y_s^n) \rangle \chi_\epsilon(s, Y_s^n) ds \\ &\quad - 2n \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), Y_s^n - \pi(s, Y_s^n) \rangle (1 - \chi_\epsilon(s, Y_s^n)) ds \\ (4.18) \quad &:= I_{31} + I_{32} + I_{33}. \end{aligned}$$

By Lemma 3.3 (i) we immediately see that $|I_{33}| \leq cn\epsilon^2$. Furthermore, using Lemma 3.4 (ii) we see that

$$(4.19) \quad |I_{32}| \leq c\sqrt{\epsilon}n \int_t^T (d(Y_s^n, D_{\epsilon,s}))^{3/2} \chi_\epsilon(s, Y_s^n) ds.$$

Using the inequality $ab \leq \frac{3a^{\frac{4}{3}}}{4} + \frac{b^4}{4}$ with $a = d(Y_s^n, D_{\epsilon,s})^{\frac{3}{2}}$ and $b = c\sqrt{\epsilon}$ we deduce from (4.19) that

$$(4.20) \quad |I_{32}| \leq cn\epsilon^2 + n \int_t^T (d(Y_s^n, D_{\epsilon,s}))^2 \chi_\epsilon(s, Y_s^n) ds.$$

Putting the estimate (4.20) into (4.18) together we can conclude that

$$I_3 \leq cn\epsilon^2 - n \int_t^T |Y_s^n - \pi_\epsilon(s, Y_s^n)|^2 \chi_\epsilon(s, Y_s^n) ds.$$

Hence, putting the estimate for I_1 and I_3 together we can conclude that

$$\begin{aligned} I_1 + I_3 &\leq c\epsilon + cn\epsilon^2 - \frac{3}{4}n \int_t^T \varphi_\epsilon(s, Y_s^n) \chi_\epsilon(s, Y_s^n) ds \\ (4.21) \quad &\quad + c \int_t^T \left(\frac{1}{n} + \epsilon \right) |f(s, Y_s^n, Z_s^n, U_s^n)|^2 ds. \end{aligned}$$

Combining (4.17) and (4.21) we have proved that

$$\begin{aligned} \varphi_\epsilon(t, Y_t^n) &\leq c(\epsilon + n\epsilon^2) - \frac{3}{4}n \int_t^T \varphi_\epsilon(s, Y_s^n) \chi_\epsilon(s, Y_s^n) ds \\ &\quad + c \int_t^T \left(\frac{1}{n} + \epsilon \right) |f(s, Y_s^n, Z_s^n, U_s^n)|^2 ds \\ &\quad - 2 \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), Z_s^n dW_s \rangle \end{aligned}$$

$$(4.22) \quad -2 \int_t^T \int_U \langle Y_s^n - \pi_\epsilon(s, Y_s^n), U_s^n(e) \rangle \mu(de, ds).$$

In particular,

$$(4.23) \quad \begin{aligned} & \varphi_\epsilon(t, Y_t^n) + \frac{3}{4}n \int_t^T \varphi_\epsilon(s, Y_s^n) \chi_\epsilon(s, Y_s^n) ds \\ & \leq c(\epsilon + n\epsilon^2) + c \int_t^T \left(\frac{1}{n} + \epsilon\right) |f(s, Y_s^n, Z_s^n, U_s^n)|^2 ds \\ & - 2 \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), Z_s^n dW_s \rangle - 2 \int_t^T \int_U \langle Y_s^n - \pi_\epsilon(s, Y_s^n), U_s^n(e) \rangle \mu(de, ds). \end{aligned}$$

where c is independent of ϵ and n . The estimate in Lemma 4.2 (ii) now follows from taking expectation in (4.23) and using the Lipschitz property of f and Lemma 4.1 (i) and (ii). Similarly, using (4.22) we can conclude that

$$\sup_{0 \leq t \leq T} E[\varphi_\epsilon(t, Y_t^n)] \leq c\left(\frac{1}{n} + \epsilon + n\epsilon^2\right),$$

for a constant c , independent of ϵ and n . Since $\varphi_\epsilon(t, Y_t) \geq 0$ for all $t \in [0, T]$ we can, repeating the arguments above, also conclude from (4.15) that

$$(4.24) \quad \begin{aligned} & E \left[\int_t^T (\partial_s \varphi_\epsilon)(s, Y_s^n) ds \right] + E \left[\int_t^T \sum_{i,j} (Z_s^n Z_s^{n,*})_{ij} \partial_{y_i y_j}^2 \varphi_\epsilon(s, Y_s) ds \right] \\ & + E \left[\int_t^T \int_U [\varphi_\epsilon(s, Y_s^n + U_s^n(e)) - \varphi_\epsilon(Y_s^n) - \langle \nabla \varphi_\epsilon(s, Y_s^n), U_s^n(e) \rangle] p(de, ds) \right] \\ & \leq c\left(\frac{1}{n} + \epsilon + n\epsilon^2\right), \end{aligned}$$

for some constant $1 \leq c < \infty$, for all $t \in [0, T]$. Once again using the Lipschitz property of f and Lemma 4.1 in (4.23) we see that to complete the proof of Lemma 4.2 (i) it remains to control the terms

$$(4.25) \quad \begin{aligned} (i) \quad & E \left[\sup_{0 \leq t \leq T} \left| \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), Z_s^n dW_s \rangle \right| \right] \\ (ii) \quad & E \left[\sup_{0 \leq t \leq T} \left| \int_t^T \int_U \langle Y_s^n - \pi_\epsilon(s, Y_s^n), U_s^n(e) \rangle \mu(de, ds) \right| \right]. \end{aligned}$$

We first treat (4.25) (i). As in (4.14) we use the Burkholder-Davis-Gundy inequality to see that

$$(4.26) \quad \begin{aligned} & E \left[\sup_{0 \leq t \leq T} \left| \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), Z_s^n dW_s \rangle \right| \right] \\ & \leq E \left[\left(\int_0^T |(Y_s^n - \pi_\epsilon(s, Y_s^n))^* Z_s^n|^2 ds \right)^{1/2} \right]. \end{aligned}$$

By the convexity of φ_ϵ and equivalence of Euclidean norms (see [GP] p. 115) we have that

$$\frac{|(Y_s^n - \pi_\epsilon(s, Y_s^n))^* Z_s^n|^2}{|(Y_s^n - \pi_\epsilon(s, Y_s^n))|^2} \mathcal{I}_{\{Y_s^n \notin D_{e,s}\}} \leq c \left(\sum_{i,j} (Z_s^n Z_s^{n,*})_{ij} \partial_{y_i y_j}^2 \varphi_\epsilon(s, Y_s) \right),$$

where \mathcal{I} is the indicator function, i.e., $\mathcal{I}_{\{Y_s^n \notin D_{\epsilon,s}\}} = 1$ if $Y_s^n \notin D_{\epsilon,s}$ and 0 otherwise. Hence, it follows from (4.24) that for $t \in [0, T]$

$$(4.27) \quad \int_t^T \frac{|(Y_s^n - \pi_\epsilon(s, Y_s^n))^* Z_s^n|^2}{|(Y_s^n - \pi_\epsilon(s, Y_s^n))|^2} \mathcal{I}_{\{Y_s^n \notin D_{\epsilon,s}\}} ds \leq c \left(\frac{1}{n} + \epsilon + n\epsilon^2 \right).$$

Note also that $|Y_s^n - \pi_\epsilon(s, Y_s^n)| = |Y_s^n - \pi_\epsilon(s, Y_s^n)| \mathcal{I}_{\{Y_s^n \notin D_{\epsilon,s}\}}$ and that

$$(4.28) \quad \begin{aligned} & \int_t^T |(Y_s^n - \pi_\epsilon(s, Y_s^n))^* Z_s^n|^2 ds \\ & \leq \sup_{t \leq s \leq T} \varphi_\epsilon(s, Y_s^n) \int_t^T \frac{|(Y_s^n - \pi_\epsilon(s, Y_s^n))^* Z_s^n|^2}{|(Y_s^n - \pi_\epsilon(s, Y_s^n))|^2} \mathcal{I}_{\{Y_s^n \notin D_{\epsilon,s}\}} ds. \end{aligned}$$

We again use $ab \leq \eta a^2 + \frac{b^2}{\eta}$ to conclude from (4.26), (4.27) and (4.28) that

$$(4.29) \quad \begin{aligned} & E \left[\sup_{0 \leq t \leq T} \left| \int_t^T \langle Y_s^n - \pi_\epsilon(s, Y_s^n), Z_s^n dW_s \rangle \right| \right] \\ & \leq \eta E \left[\sup_{0 \leq t \leq T} \varphi_\epsilon(t, Y_t^n) \right] + c_\eta \left(\frac{1}{n} + \epsilon + n\epsilon^2 \right) \end{aligned}$$

for some constant $c_\eta < \infty$ depending on the degree of freedom $\eta > 0$.

We now treat the term (4.25) (ii). Using Taylor's theorem we see that

$$\begin{aligned} E \left[\int_t^T \int_U [\varphi_\epsilon(s, Y_s^n + U_s^n(e)) - \varphi_\epsilon(Y_s^n) - \langle \nabla \varphi_\epsilon(s, Y_s^n), U_s^n(e) \rangle] p(de, ds) \right] \\ \geq \check{c} E \left[\int_t^T |U(e)|^2 \lambda(de) ds \right], \end{aligned}$$

for some constant $\check{c} \geq 0$. Furthermore, by the strong convexity of φ_ϵ , and this is a consequence of (2.2), there exists a constant $\kappa > 0$ such that $\check{c} \geq \kappa > 0$. Therefore we can, in a way similar to the above, conclude that

$$(4.30) \quad \begin{aligned} & E \left[\sup_{0 \leq t \leq T} \left| \int_t^T \int_U \langle Y_s^n - \pi_\epsilon(s, Y_s^n), U_s^n(e) \rangle \mu(de, ds) \right| \right] \\ & \leq E \left[\left(\int_t^T \int_U |Y_s^n - \pi_\epsilon(s, Y_s^n)|^2 |U_s^n(e)|^2 \lambda(de) ds \right)^{1/2} \right] \\ & \leq E \left[\left(\sup_{t \leq s \leq T} \varphi_\epsilon(s, Y_s^n) \int_t^T \int_U |U_s^n(e)|^2 \lambda(de) ds \right)^{1/2} \right] \\ & \leq \eta E \left[\sup_{0 \leq t \leq T} \varphi_\epsilon(t, Y_t^n) \right] + c_\eta \left(\frac{1}{n} + \epsilon + n\epsilon^2 \right). \end{aligned}$$

Once again, $c_\eta < \infty$ is a constant depending on the degree of freedom $\eta > 0$.

The proof of Lemma 4.2 is now completed by choosing η small enough (in analogy with (4.8), (4.9)) and combining (4.23) with (4.29), (4.30). \square

Lemma 4.3. *Let $D \subset \mathbb{R}^{d+1}$ be a time-dependent domain satisfying (2.2) - (2.5). Then there exists c , $1 \leq c < \infty$, independent of n such that*

$$\begin{aligned} (i) \quad & E \left[\sup_{0 \leq t \leq T} (d(Y_t^n, D_t))^2 \right] \leq \frac{c}{n}, \\ (ii) \quad & E \left[\int_0^T (d(Y_t^n, D_t))^2 dt \right] \leq \frac{c}{n^2}, \end{aligned}$$

whenever $n \geq 1$.

Proof. Let, for $\epsilon > 0$ small, D_ϵ be as in Lemma 3.2. Then, using Lemma 3.2 we have

$$h(D_t, D_{\epsilon,t}) < \epsilon \text{ for all } t \in [0, T].$$

Hence,

$$d(Y_t^n, D_t) \leq d(Y_t^n, D_{\epsilon,t}) + \epsilon \text{ for all } t \in [0, T].$$

Applying Lemma 4.2 and letting $\epsilon \rightarrow 0$ completes the proof. \square

4.4. (Y_t^n, Z_t^n, U_t^n) is a Cauchy sequence.

Lemma 4.4. *There exists a constant c such that the following holds whenever $m, n \in \mathbb{Z}^+$:*

$$\begin{aligned} E \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 + \int_0^t \|Z_s^n - Z_s^m\|^2 ds \right] &\leq c \left(\frac{1}{n} + \frac{1}{m} \right), \\ E \left[\int_0^T \int_U |U_s^n(e) - U_s^m(e)|^2 \lambda(de) ds \right] &\leq c \left(\frac{1}{n} + \frac{1}{m} \right). \end{aligned}$$

Proof. Applying Ito's formula to $|Y_t^n - Y_t^m|^2$ we get that

$$\begin{aligned} &|Y_t^n - Y_t^m|^2 + \int_t^T \|Z_s^n - Z_s^m\|^2 ds \\ &+ \int_t^T \int_U |U_s^n(e) - U_s^m(e)|^2 p(de, ds) \\ = &2 \int_t^T \langle Y_s^n - Y_s^m, f(s, Y_s^n, Z_s^n, U_s^n) - f(s, Y_s^m, Z_s^m, U_s^m) \rangle ds \\ &- 2 \int_t^T \langle Y_s^n - Y_s^m, (Z_s^n - Z_s^m) dW_s \rangle \\ &- 2 \int_t^T \int_U \langle Y_s^n - Y_s^m, U_s^n(e) - U_s^m(e) \rangle \mu(de, ds) \\ &- 2n \int_t^T \langle Y_s^n - Y_s^m, Y_s^n - \pi(s, Y_s^n) \rangle ds \\ (4.31) \quad &+ 2m \int_t^T \langle Y_s^n - Y_s^m, Y_s^m - \pi(s, Y_s^m) \rangle ds. \end{aligned}$$

Hence, taking expectation and using the Lipschitz character of f we deduce that

$$\begin{aligned} &E[|Y_t^n - Y_t^m|^2] + E \left[\int_t^T \|Z_s^n - Z_s^m\|^2 ds \right] \\ &+ E \left[\int_t^T \int_U |U_s^n(e) - U_s^m(e)|^2 \lambda(de) ds \right] \\ \leq &cE \left[\int_t^T (|Y_s^n - Y_s^m|^2 + |Y_s^n - Y_s^m| \|Z_s^n - Z_s^m\|) ds \right] \\ &+ cE \left[\int_t^T |Y_s^n - Y_s^m| \int_U |U_s^n(e) - U_s^m(e)| \lambda(de) ds \right] \\ (4.32) \quad &- 2E \left[\int_t^T \langle Y_s^n - Y_s^m, n(Y_s^n - \pi(s, Y_s^n)) - m(Y_s^m - \pi(s, Y_s^m)) \rangle ds \right]. \end{aligned}$$

Note that

$$\begin{aligned} & -\langle Y_s^n - Y_s^m, n(Y_s^n - \pi(s, Y_s^n)) - m(Y_s^m - \pi(s, Y_s^m)) \rangle \\ &= \langle Y_s^n - Y_s^m, n(Y_s^n - \pi(s, Y_s^n)) \rangle + \langle Y_s^n - Y_s^m, m(Y_s^m - \pi(s, Y_s^m)) \rangle. \end{aligned}$$

Using Lemma 3.1 (ii) we have that

$$\begin{aligned} \langle Y_s^m - Y_s^n, n(Y_s^n - \pi(s, Y_s^n)) \rangle &\leq n \langle Y_s^m - \pi(s, Y_s^m), Y_s^n - \pi(s, Y_s^n) \rangle \\ \langle Y_s^n - Y_s^m, m(Y_s^m - \pi(s, Y_s^m)) \rangle &\leq m \langle Y_s^n - \pi(s, Y_s^n), Y_s^m - \pi(s, Y_s^m) \rangle. \end{aligned}$$

Furthermore,

$$\begin{aligned} & 2E \left[\int_t^T \langle Y_s^m - Y_s^n, n(Y_s^n - \pi(s, Y_s^n)) \rangle ds \right] \\ &\leq 2nE \left[\int_t^T |Y_s^m - \pi(s, Y_s^m)| |Y_s^n - \pi(s, Y_s^n)| ds \right] \\ &\leq nE \left[\int_t^T \beta (d(Y_s^m, D_s))^2 + \beta^{-1} (d(Y_s^n, D_s))^2 ds \right] \\ &\leq c(n\beta m^{-2} + \beta^{-1}n^{-1}) \leq cm^{-1} \end{aligned}$$

where we have used Lemma 4.3 (ii) and chosen the degree of freedom to equal $\beta = m/n$. This argument can be repeated with $n \langle Y_s^m - Y_s^n, n(Y_s^n - \pi(s, Y_s^n)) \rangle$ replaced by $m \langle Y_s^n - \pi(s, Y_s^n), Y_s^m - \pi(s, Y_s^m) \rangle$ resulting in the bound cn^{-1} . Put together we can conclude that

$$\begin{aligned} & -2E \left[\int_t^T \langle Y_s^n - Y_s^m, n(Y_s^n - \pi(s, Y_s^n)) - m(Y_s^m - \pi(s, Y_s^m)) \rangle ds \right] \\ (4.33) \quad & \leq c(n^{-1} + m^{-1}). \end{aligned}$$

Combining (4.32), (4.33) and using Cauchy's inequality as in (4.8), (4.9) we can conclude that

$$\begin{aligned} & E[|Y_t^n - Y_t^m|^2] + \frac{1}{2}E \left[\int_t^T \|Z_s^n - Z_s^m\|^2 ds \right] \\ & + \frac{1}{2}E \left[\int_t^T \int_U |U_s^n(e) - U_s^m(e)|^2 \lambda(de) ds \right] \\ (4.34) \quad & \leq cE \left[\int_t^T |Y_s^n - Y_s^m|^2 ds \right] + c(n^{-1} + m^{-1}) \end{aligned}$$

where c is independent of n and m . By Gronwall's inequality we then have, using (4.34),

$$(4.35) \quad E[|Y_t^n - Y_t^m|^2] \leq c(n^{-1} + m^{-1}).$$

Subsequently,

$$\begin{aligned} & E \left[\int_t^T \|Z_s^n - Z_s^m\|^2 ds \right] \leq c(n^{-1} + m^{-1}) \\ (4.36) \quad & E \left[\int_t^T \int_U |U_s^n(e) - U_s^m(e)|^2 \lambda(de) ds \right] \leq c(n^{-1} + m^{-1}). \end{aligned}$$

Using (4.35), (4.36), Lemma 4.3 (i), and now familiar arguments based on the Burkholder-Davis-Gundy inequality, we can, starting from (4.31), also deduce that

$$E \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 \right] \leq c \left(\frac{1}{n} + \frac{1}{m} \right),$$

to complete the proof of Lemma 4.4. We omit further details. \square

5. THE FINAL ARGUMENT: PROOF OF THEOREM 2.1

Using Lemma 4.4 we can conclude that (Y_t^n, Z_t^n, U_t^n) is a Cauchy sequence in the space of progressively measurable processes (Y_t, Z_t, U_t) satisfying

$$E \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] + E \left[\int_0^T \|Z_t\|^2 ds \right] + E \left[\int_0^T \int_U |U_t(e)|^2 \lambda(de) ds \right] < \infty.$$

Hence, taking a subsequence if necessary, we have a sequence $(Y_t^n, Z_t^n, U_t^n)_{n \geq 0}$ and a triple of processes (Y_t, Z_t, U_t) such that

$$Y_t = \lim_{n \rightarrow \infty} Y_t^n, \quad Z_t = \lim_{n \rightarrow \infty} Z_t^n, \quad U_t = \lim_{n \rightarrow \infty} U_t^n$$

in the sense that

$$\begin{aligned} (i) \quad & E \left[\sup_{0 \leq t \leq T} |Y_t^n - Y_t|^2 \right] \rightarrow 0, \\ (ii) \quad & E \left[\int_0^T \|Z_t^n - Z_t\|^2 ds \right] \rightarrow 0, \\ (5.1) \quad (iii) \quad & E \left[\int_0^T \int_U |U_t^n(e) - U_t(e)|^2 \lambda(de) ds \right] \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Furthermore, by Lemma 4.1 we have

$$\begin{aligned} (i) \quad & E \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] < \infty, \\ (5.2) \quad (ii) \quad & E \left[\int_0^T \|Z_t\|^2 dt + \int_0^T \int_U |U_s(e)|^2 \lambda(de) ds \right] < \infty. \end{aligned}$$

Note that, as a uniform limit of càdlàg functions $\{Y_t^n\}$, we immediately have that $Y_t \in \mathcal{D}([0, T], \mathbb{R}^d)$ and, by Lemma 4.3 (i), we have $Y_t \in \overline{D}$. Recall that

$$\Lambda_t^n = -n \int_0^t (Y_s^n - \pi(s, Y_s^n)) ds = \int_0^t -\frac{(Y_s^n - \pi(s, Y_s^n))}{|Y_s^n - \pi(s, Y_s^n)|} d|\Lambda^n|_s$$

and that $-(Y_s^n - \pi(s, Y_s^n))/|Y_s^n - \pi(s, Y_s^n)|$ is an element in the inward directed normal cone to D_s at $\pi(s, Y_s^n) \in \partial D_s$. Using (4.2) (iii), (5.1) and (5.2) it follows that there exists Λ_t such that

$$\Lambda_t = \lim_{n \rightarrow \infty} \Lambda_t^n = \lim_{n \rightarrow \infty} -n \int_0^t (Y_s^n - \pi(s, Y_s^n)) ds$$

in the sense that

$$E \left[\sup_{0 \leq t \leq T} |\Lambda_t^n - \Lambda_t|^2 \right] \rightarrow 0.$$

Hence, as $(\Lambda_t^n(\omega))_{0 \leq t \leq T}$ is continuous, $(\Lambda_t(\omega))_{0 \leq t \leq T}$ is continuous in t a.s.

5.1. Existence: $(Y_t, Z_t, U_t, \Lambda_t)$ **is a solution.** We will now prove that the constructed quadruple $(Y_t, Z_t, U_t, \Lambda_t)$ is a solution to our original problem. We first note that, as a limit of (Y_t^n, Z_t^n, U_t^n) , (Y_t, Z_t, U_t) are progressively measurable, $Y_t \in \mathcal{D}([0, T], \mathbb{R}^d)$ and Z and U are predictable. Hence it remains to verify that $(Y_t, Z_t, U_t, \Lambda_t)$ satisfies (i)-(vi) stated in Definition 1 and that $\Lambda \in \mathcal{BV}([0, T], \mathbb{R}^d)$. That $(Y_t, Z_t, U_t, \Lambda_t)$ satisfies (i)-(iii) was proved above and (iv) is a consequence of Lemma 4.3 and (5.1) (i). Hence we in the following focus on properties (v) and (vi). As mentioned above, we have that $(\Lambda_t(\omega))_{0 \leq t \leq T}$ is continuous in t for almost all ω by uniform convergence. Furthermore, using that

$$\int_t^T d|\Lambda^n|_s = n \int_t^T |Y_s^n - \pi(s, Y_s^n)| ds$$

we see from Lemma 4.1 (iii) that

$$E \left[\int_0^T d|\Lambda^n|_s \right] = E \left[n \int_0^T |Y_s^n - \pi(s, Y_s^n)| ds \right] \leq c \text{ for all } n \in \mathbb{Z}_+,$$

for some constant c which is independent of n . It follows that $\Lambda_t(\omega)$ is of bounded total variation on $[0, T]$ for almost all ω . Hence, it only remains to verify that

$$(v) \quad \Lambda_t = \int_0^t \gamma_s d|\Lambda|_s, \quad \gamma_s \in N_s^1(Y_s) \text{ whenever } Y_s \in \partial D_s,$$

$$(5.3) \quad (vi) \quad d|\Lambda|(\{t \in [0, T] : (t, Y_t) \in D\}) = 0.$$

To verify the statements in (5.3) we will use the following lemma.

Lemma 5.1. *Let $\{\Lambda^n\}_{n \in \mathbb{Z}_+}$ be a sequence of continuous functions, $\Lambda^n : [0, T] \rightarrow \mathbb{R}^d$, which converges uniformly to Λ as $n \rightarrow \infty$. Assume $\Lambda^n \in \mathcal{BV}([0, T], \mathbb{R}^d)$ and that $|\Lambda^n|_T \leq c$, for some $c < \infty$, hold for all n . Let $\{f^n\}_{n \in \mathbb{Z}_+}$ be a sequence of càdlàg functions, $f^n : [0, T] \rightarrow \mathbb{R}^d$, converging uniformly to f as $n \rightarrow \infty$. Then,*

$$\lim_{n \rightarrow \infty} \int_0^t \langle f_s^n, d\Lambda_s^n \rangle = \int_0^t \langle f_s, d\Lambda_s \rangle$$

for all $t \in [0, T]$.

Proof. This is essentially Lemma 5.8 in [GP], see also [S]. □

Using Lemma 3.1 (i) we see that ,

$$\langle z_t - Y_t^n, Y_t^n - \pi(t, Y_t^n) \rangle \leq 0$$

for any càdlàg process z_t taking values in $\overline{D_t}$. Hence, for any such process z_t we have that

$$\int_0^t -n \langle z_s - Y_s^n, Y_s^n - \pi(s, Y_s^n) \rangle ds = \int_0^t \langle z_s - Y_s^n, d\Lambda_s^n \rangle \geq 0.$$

Passing to the limit we obtain, using Lemma 5.1, that

$$(5.4) \quad \int_0^t \langle Y_s - z_s, d\Lambda_s \rangle \leq 0$$

for all $z \in \mathcal{D}([0, T], \mathbb{R}^d)$ taking values in \overline{D} , and for all $t \in [0, T]$. Next, let $\tau \in [0, T)$ be any time such that $Y_\tau \in D$ and let $\hat{\gamma}$ be a unit vector in \mathbb{R}^d . Since Y_s is right-continuous, taking assumption (2.4) into account, we see that there exists $\epsilon > 0$ and $\delta > 0$ such that $Y_s + \epsilon \hat{\gamma} \in D$ and $Y_s - \epsilon \hat{\gamma} \in D$ whenever $s \in [\tau, \tau + \delta]$. However, this in combination with (5.4) implies that

$$0 \leq \int_\tau^{\tau+\delta} \hat{\gamma} d\Lambda_t \leq 0,$$

which in turn implies (vi) in (5.3). Hence

$$\Lambda_t = \int_0^t \gamma_s d|\Lambda|_s,$$

for some vector field $\gamma_s \in \mathbb{R}^d$, with support on ∂D , and such that $|\gamma_s| = 1$ (see (2.6)). To conclude the existence part of Theorem 2.1 it remains to show (5.3) (v), i.e., that $\gamma_s \in N_s^1(Y_s)$ whenever $Y_s \in \partial D_s$. However, using the above and (2.6) we see that to prove $\gamma_s \in N_s^1(Y_s)$ whenever $Y_s \in \partial D_s$, it is enough to prove that if $Y_s \in \partial D_s$ and if $\langle Y_s - z_s, \gamma_s \rangle \leq 0$ for all $z_s \in D_s$, then $\gamma_s \in N_s(Y_s)$. To do this, take $\beta = Y_s - \gamma_s \in \mathbb{R}^d$. Then,

$$|\beta - z_s|^2 = |\beta - Y_s|^2 + |Y_s - z_s|^2 + 2\langle \beta - Y_s, Y_s - z_s \rangle$$

for all $z_s \in D_s$. Hence, if $\langle \beta - Y_s, Y_s - z_s \rangle = -\langle Y_s - z_s, \gamma_s \rangle \geq 0$, then we have that

$$|\beta - z_s|^2 \geq |\beta - Y_s|^2$$

for all $z_s \in D_s$, which implies $\gamma_s \in N_s(Y_s)$. This proves (5.3) (v) and thus the proof of the existence part of Theorem 2.1 is complete. \square

5.2. Uniqueness: $(Y_t, Z_t, U_t, \Lambda_t)$ is the only solution. We here prove the uniqueness part of Theorem 2.1 using Ito's formula. Indeed, assume that $(Y^i, Z^i, U^i, \Lambda^i)$, $i = 1, 2$, are two solutions to the reflected BSDE under consideration and define

$$\{\Delta Y_t, \Delta Z_t, \Delta U_t, \Delta \Lambda_t\} = \{Y_t^1 - Y_t^2, Z_t^1 - Z_t^2, U_t^1 - U_t^2, \Lambda_t^1 - \Lambda_t^2\}.$$

Then, applying Ito's formula to $|\Delta Y_t|^2$ and taking expectation we have that

$$\begin{aligned} & E \left[|\Delta Y_t|^2 + \int_t^T |\Delta Z_s|^2 ds + \int_t^T \int_U |\Delta U_s(e)|^2 \lambda(de) ds \right] \\ &= 2E \left[\int_t^T \langle \Delta Y_s, f(s, Y_s^1, Z_s^1, U_s^1) - f(s, Y_s^2, Z_s^2, U_s^2) \rangle ds \right] \\ &+ 2E \left[\int_t^T \langle \Delta Y_s, d\Delta \Lambda_s \rangle \right]. \end{aligned}$$

Using (5.4) we see that the last term on the right hand side in the above display is ≤ 0 . Next, using the Lipschitz character of f and standard manipulations, see (4.7), (4.8), we can conclude that

$$\begin{aligned} & E \left[|\Delta Y_t|^2 + \int_t^T |\Delta Z_s|^2 ds + \int_t^T \int_U |\Delta U_s(e)|^2 \lambda(de) ds \right] \\ &\leq cE \left[\int_t^T |\Delta Y_s|^2 ds + \frac{1}{2} \int_t^T |Z_s|^2 ds + \frac{1}{2} \int_t^T |\Delta U_s(e)|^2 \lambda(de) ds \right]. \end{aligned}$$

Applying Gronwall's lemma we see that $\{\Delta Y_t, \Delta Z_t, \Delta U_t\}$ must be identically zero a.s. By (iii) of Definition 1, the same applies to $\Delta \Lambda_t$ and the proof is hence complete. \square

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